Section 3.1: Vector Spaces

A **vector space over** $\mathbb{R}$ is a set V of objects (called **vectors**), together with two operations, addition and scalar multiplication, which satisfy the following:

1. V is closed under addition.
2. V is closed under scalar multiplcation.
3. For all $\mathbf{x}, \mathbf{y} \in V$, we have $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.
4. For all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, we have $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$.
5. There exists $\mathbf{0} \in V$ such that for all $\mathbf{x} \in V$, we have $\mathbf{x} + \mathbf{0} = \mathbf{x}$. (The vector $\mathbf{0}$ is called a **zero vector** for V.)
6. For each $\mathbf{x} \in V$, there exists $\mathbf{y} \in V$ such that $\mathbf{x} + \mathbf{y} = \mathbf{0}$. ($\mathbf{y}$ is called an **additive inverse** of $\mathbf{x}$.)
7. For all $\mathbf{x} \in V$, we have $1\mathbf{x} = \mathbf{x}$.
8. For all $\alpha, \beta \in \mathbb{R}$ and all $\mathbf{x} \in V$, we have $(\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x})$.
9. For all $\alpha \in \mathbb{R}$ and all $\mathbf{x}, \mathbf{y} \in V$, we have $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$.
10. For all $\alpha, \beta \in \mathbb{R}$ and all $\mathbf{x} \in V$, we have $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$.

Examples of Vector Spaces

1. For all n, $\mathbb{R}^n$ is a vector space.
2. For all m, n, $M_{m,n}(\mathbb{R})$ is a vector space.
3. The set $\mathcal{P}(\mathbb{R})$ of all polynomials in one variable x with real coefficients is a vector space.
4. The set $\mathcal{P}_n(\mathbb{R})$ of all polynomials of degree at most n in one variable x with real coefficients is a vector space.
5. The set $\mathcal{F}(\mathbb{R})$ of all functions from $\mathbb{R}$ to $\mathbb{R}$ is a vector space.

Properties of Vector Spaces

Theorem 3.7: Let V be a vector space.

1. The zero vector $\mathbf{0}$ is unique.
2. Given $\mathbf{x} \in V$, its additive inverse is unique.
3. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$. If $\mathbf{x} + \mathbf{z} = \mathbf{y} + \mathbf{z}$, then $\mathbf{x} = \mathbf{y}$.
4. For all $\mathbf{x} \in V$, we have $0\mathbf{x} = \mathbf{0}$.
5. For all $\alpha \in \mathbb{R}$, we have $\alpha\mathbf{0} = \mathbf{0}$.
6. For all $\mathbf{x} \in V$, the vector $(-1)\mathbf{x}$ is the additive inverse of $\mathbf{x}$.